# Lecture Notes 2

Tom Flesher

February 7, 2015

# Chapter 2

# Revenue, Cost, and Profit

## 2.1 Revenue

One of the ways to describe the financial health of a firm is its total revenue. Revenue is the flow of money into a firm due to sales of goods. When a firm considers its financial health, one of the key measures it is revenue, due to the importance of money for creating continued growth in income. Revenue is sometimes thought of as "the gross," or the income that you count before thinking about costs.

#### The Revenue Function

Total revenue is defined as $P \times Q$. If we hold price constant, then the raising quantity will raise revenue; if we hold quantity constant, raising price will raise revenue. In reality, however, quantity supplied increases in price and quantity demanded decreases in price, so it can be difficult to determine the exact change in revenue for any small change in price without more information. The tool of price-elasticity was designed to solve this problem and will be the focus of later discussion.

Since revenue depends on price, and price depends on demand, revenue usually requires us to know a demand correspondence before solving for it. When solving for revenue, solve the demand equation for price and substitute into the revenue formula such that only the variable $Q$ remains.

### 2.1.1 Revenue Maximization: A Worked Example

**Problem.** Suppose a revenue-maximizing firm faces demand $Q^D = 200 - 2P$. What quantity should it produce? What is the firm's maximum revenue?

**Solution.** Recall that $TR = P \times Q$. Solve the demand equation for $P$:

$$Q = 200 - 2P \rightarrow 2P = 200 - Q \rightarrow P = 100 - \frac{1}{2}Q$$

Now, substitute $P = 100 - \frac{1}{2}Q$ into the revenue formula, yielding $TR = Q(100 - \frac{1}{2}Q)$. Distributing Q, this is equivalent to $TR = 100Q - \frac{1}{2}Q^2$, yielding a straightforward quadratic.

To maximize profit, take the first derivative of total revenue, called *marginal revenue*[1]: $MR = 100 - Q$. At the point where $MR = 0$, revenue is maximized. Solving $0 = 100 - Q$ yields $Q = 100$, meaning that $Q = 100$ is a stationary point. Since revenue's second derivative is negative ($MR' = TR'' = -1$), we know this is a maximum. Revenue is maximized at $Q = 100$.

If $Q = 100$, then $P = 100 - \frac{1}{2}(100) = 100 - 50 = 50$. Revenue where $P = 100$ and $Q = 50$ is $TR = 500$.

### 2.1.2 Price and Output Effects

Assume for now that a firm's can supply as many units as it wishes and that it faces a market with a downward-sloping demand curve. What happens if the firm lowers the price of its good? Call the first price and quantity pair $(Q_0, P_0)$ and the second pair $(Q_1, P_1)$. Because quantity demanded has an inverse relationship with price, lowering the price will make existing buyers happier and entice more people to buy. Thus, when $P_1 < P_0$, $Q_1 > Q_0$. We can then sort buyers into two groups: those who would have purchased even at the higher price and those who would not.

Two things happen when the firm lowers its price, each affecting one of those groups. The **price effect** measures how much revenue is lost from the first group. Because those buyers would have purchased even at the higher price, the firm cannot capture their entire willingness to pay. As a result, some revenue is lost from this group. That group, because it considers only the people who would have bought at the higher price, has $Q_0$ members, and revenue shrinks by the difference between prices $(P_1 - P_0)$ for each member. Thus, the price effect equals $Q_0(P_1 - P_0)$.

The **output effect**, on the other hand, measures how much revenue is gained from the first group. Because we gain $Q_1 - Q_0$ buyers, each paying $P_1$, the amount of new revenue gained is $P_1(Q_1 - Q_0)$. Note that if the price rises rather than falls, the price effect will be a gain in revenue and the output effect will be a loss of revenue.

## 2.2 Elasticity

How responsive is consumer behavior to price? The economic concept of **elasticity** is designed to answer that question. When economists say "elasticity" they usually mean the **price-elasticity of demand**, which measures how sensitive purchasing decisions are to changes in price.

[1] In Economics, "marginal" always means "the first derivative of total."

Consider, for example, the responsiveness of drivers to changes in the price of gasoline. Many people do a relatively fixed amount of driving, regardless of the price of gasoline. If the price of gas, which has miraculously dropped to $2.19 in Stony Brook, suddenly doubled, very few people would be likely to drop their classes to avoid driving to campus. Demand for gasoline is *price-inelastic.*

On the other hand, consider what might happen if the price of eggs tripled. Because you can quickly substitute other goods for eggs (e.g. cereal and milk), a spike in the price of eggs would lead to a large decrease in the number of eggs consumed. Because of the large consumer response to price changes, demand for eggs is *price-elastic.*

The formula for price-elasticity of demand is $\frac{\%\Delta Q^D}{\%\Delta P}$, the ratio of percentage change in quantity demanded to percentage change in price. This will almost always be negative, since an increase (positive change) in price will yield a decrease (negative change) in quantity demanded. Economists consider only the absolute value of the price-elasticity of demand - in other words, drop the negative sign.

When the price-elasticity of demand has a value greater than 1, demand is elastic. That means that a 1% change in price will create more than a 1% change in quantity demanded, meaning we can expect raising the price to lower revenue.

When the price-elasticity of demand has a value less than 1, demand is inelastic. A 1% change in price will lead to less than a 1% change in quantity demanded, meaning that raising the price a little will lead to an increase in revenue.

When the price-elasticity of demand has a value of exactly 1, demand is unit-elastic. A 1% change in price corresponds to a 1% change in quantity demanded, meaning that raising the price has no effect on revenue.

### 2.2.1 The Elasticity Rule

The **elasticity rule** helps a firm predict what will happen if it adjusts the price of its goods. It states a relationship between elasticity and predicted changes in revenue. In the case of:

$$\begin{cases} \textbf{Elastic Demand}, & \text{Raising the price of the good will lower revenue;} \\ \textbf{Inelastic Demand}, & \text{Raising the price of the good will raise revenue;} \\ \textbf{Unit-Elastic Demand}, & \text{Price changes have no effect on revenue.} \end{cases}$$

## 2.3 Cost

Total cost is the negative portion of the profit function $\Pi(Q, P)$. These costs can be sorted into two different types: **fixed costs** and **variable costs**. The nature of specific cost types will determine how a firm must account for those costs in optimizing its production.

The cost function does not depend on price. (We'll see later on that price is actually an implicit function of costs.) The cost function varies only with quantity, since we treat fixed costs as a parameter and variable costs as either a parameter or a function of quantity. Hence, though we could do a comparative statics analysis to determine the sensitivity of the cost function to adjustments in costs, the most common form of the cost function will be:

$$TC(Q) = VC + FC$$

Fixed costs are costs that are unavoidable. Examples of fixed costs include rent payments, the salaries of workers under contract, and many taxes. Fixed costs are easy to account for - they never change. If the firm is under a lease, it is obligated to pay a certain rent payment each month regardless of whether or not it chooses to produce. Similarly, salaried workers are paid whether they work or not, and consequently their salaries cannot be avoided. As such, only firms concerned with the short run need consider fixed costs, since the long run is defined as the period over which all costs are variable. Mathematically, the fixed cost is a constant added to the end of the cost function, such that the total cost at 0 quantity is equal to the fixed cost ($TC(0) = FC$).

Because some costs are paid even when not producing, the size of the fixed cost can be a large proportion of total costs. For example, a widget factory produces widgets as output while using labor, electricity, machinery, and widget materials as input. However, the factory cannot operate without a building to house it, a manager to staff the shift and plan production, and a certain base level of electrical and heating (or cooling) input to make sure the factory is habitable for its workers. That base level constitutes the fixed cost, and this fixed cost will be the majority of costs for many levels of production.

Variable costs are costs that depend on the quantity produced. It is easiest to think of material costs as the actual raw materials that are used to produce a good. Keep in mind that running a machine for a longer period of time in order to produce more goods will also add to material costs. In addition, an operator must be present to run the machine, and his labor must be accounted for in the variable cost function. Finally, many pieces of equipment will require extra maintenance when used more intensively, and as a result many cost functions include additional maintenance costs.

Despite what economists and engineers might think, when managers ask about the cost to produce a widget, they are not especially interested in the mathematical properties of a cost function. Most often, they care about one of two numbers: the **total variable cost** or the **average variable cost**. Total variable cost is simply what we refer to as $TC$ – the amount of total cost due to the variable cost ($VC = TC - FC$). Average variable cost is precisely what it sounds like - the sum of variable costs divided by the number of units produced ($AVC = \frac{VC}{Q}$). Average variable cost is a useful measure because it avoids the necessity of explaining (again) to management the properties of cost curves. When a manager asks, "What did this unit cost to produce?", answering with the average variable cost allows him to understand quickly what the relevant costs are without burdening him with unnecessary information.

The simplest form of a cost function is known as a *constant average-variable-cost function.* This function requires making some assumptions, the eponymous constant-AVC assumption being the most important. In a constant-AVC model, we assume that every unit costs exactly the same to produce. This is realistic under most conditions. For example, a small firm may not have the market power to negotiate volume discounts, so material costs for the same good will not change drastically depending on the size of a production run. Most goods manufactured are produced using an automated process, so the number of hours per unit produced will not vary greatly. Labor costs will be affected if the same operator runs into overtime pay, but this can be mitigated by operating in shifts. The constant-AVC function, then, closely models an assembly-line process in which workers are more or less fungible and material costs are more or less constant. A constant-AVC function always takes the form $VC(Q) = AVC \times Q$.

Relaxing some of these assumptions will lead to differences in the shape of the cost function. Most of the time, economists assume that costs are rising faster than quantity. This is the case in an *increasing-cost industry*, such as the market for commodities. If it is possible to "corner" a market by buying a significant amount of its supply, then costs will rise exponentially. These conditions also arise when it becomes more expensive or more difficult to extract resources, as in the market for fossil fuels such as oil. In this case, variable costs will be a convex function of Q, such as $VC(Q) = Q^2 + aQ$. This is the typical case, although most firms operate along a small-enough portion of the cost curve that their cost function is locally constant.

On the other hand, if costs rise more slowly than quantity, economists call this a *decreasing-cost industry*. An example of a decreasing-cost industry would be the harvesting of electricity from wind power. Once the initial cost of installing a windmill has been spent, each kilowatt-hour of electricity is effectively free, so the average variable cost is decreasing. Even in less extreme cases of decreasing average variable costs, costs will be a concave function of Q, such as $VC(Q) = \sqrt{Q} + bQ$.

## 2.4 Profit

**Profit** is the most common objective that a firm will attempt to maximize. Although revenue is a strong measure of sales, profit takes into account the overall flows of money into and out of the firm. Profit is defined as total revenue minus total cost, and is thus often called "the net" (in contrast to revenue as "the gross").

A firm's profit depends not only on the revenue (units sold) but also the firm's material costs (both variable and fixed costs). If a firm wishes to increase profit, then, it can try to do two things - increase revenue or decrease costs. Increasing revenue is often achieved using advertising, product differentiation, or elasticity-based price changes. Material costs can be lowered through negotiation with providers or volume discounts. Labor costs can be reduced through hiring of cheaper labor or through training to make workers more effective. Note that these two solutions can seem contradictory in the short run but can be combined in the long run; we'll see an example later on of how to determine which solution is appropriate.

### 2.4.1 Profit Maximization: A Walk-Through Guide

A firm's optimal quantity and price can be found using standard optimization techniques. Begin with a standard profit function:

$$\Pi(P, Q) = TR - TC$$

To find the maximum of any function, find its critical points using the first-derivative test. Limit those critical points to local maxima using the second-derivative test. Finally, evaluate the function at each local maximum; the highest-valued point is the global maximum.

Since profit is a linear combination of functions, its derivative with respect to quantity is a linear combination of those functions' derivatives with respect to quantity:

$$\frac{d\Pi}{dQ} = \frac{dTR}{dQ} - \frac{dTC}{dQ}$$

Economists call first derivatives "marginal" functions. The first derivative of cost is called marginal cost, for example, and the first derivative of revenue is called marginal revenue. Using the economic terminology, then, marginal profit equals marginal revenue less marginal cost:

$$M\Pi = MR - MC$$

The first derivative test tells us that at any local maximum (or local minimum, or stationary point), the first derivative of the objective function with respect to the control variable (in this case, $\frac{d\Pi}{dQ}$) is 0. As a result, we know that:

$$0 = MR - MC \rightarrow MR = MC$$

At maximum profit, marginal revenue equals marginal cost. This is because marginal revenue tells us how much additional revenue is made from selling one more unit, while marginal cost measures the production cost of that unit. If marginal revenue is greater than marginal cost, additional units could be produced, even if marginal cost is rising and marginal cost is falling. If marginal revenue is less than marginal cost, the firm "lost money" producing that unit because it cost more than could be recouped through sale.

Ensure this point is a maximum using the second derivative test, which tells us that at a local maximum, the second derivative of the objective function with respect to the control variable (in this case, $\frac{d^2\Pi}{dQ^2}$) is zero or negative:

$$\frac{d^2\Pi}{dQ^2} \leq 0 \rightarrow \frac{d^2TR}{dQ^2} \leq \frac{d^2TC}{dQ^2} \rightarrow \frac{dMR}{dQ} \leq \frac{dMC}{dQ}$$

Most functions we work with will have positive first derivatives, at least in the areas of interest. Note that we have some shortcuts available here. If revenue is linear or concave, its second derivative is always 0 or less. Similarly, since total cost is variable cost plus a constant, total cost and variable cost will have the same derivatives, and if variable cost is linear or convex, its second derivative is always 0 or more. Thus, the only time we need to use the second-derivative test is the unusual case of a concave cost function and/or a convex revenue function.

### 2.4.2 A Worked Example

**Problem.** A firm faces a market demand correspondence of $Q = 375 - 5P$ and has a constant average variable cost of 50 per unit. Fixed costs are 1000. Find the firm's profit-maximizing price and quantity. As a project engineer, what recommendations would you make about this project?

**Solution.** The firm's total cost function is $TC(Q) = 50Q + 5000$. The firm's demand, solved for $P$, is $P = 75 - \frac{1}{5}Q$, implying that total revenue is $TR = Q(75 - \frac{1}{5}Q) = 75Q - \frac{1}{5}Q^2$. Thus, the firm's total profit is:

$$\Pi(Q) = TR - TC = 75Q - \frac{1}{5}Q^2 - 50Q - 5000 = -\frac{1}{5}Q^2 + 25Q - 5000$$

Taking the first derivative and finding critical points,

$$\frac{d\Pi}{dQ} = -\frac{2}{5}Q + 25$$

$$0 = -\frac{2}{5}Q + 25 \rightarrow \frac{2}{5}Q = 125 = Q = \frac{5}{2}25 = 62.5$$

Thus, a critical point occurs at 62.5 units. Is this point a minimum, maximum, or stationary point? Taking the second derivative,

$$\frac{d^2\Pi}{dQ^2} = -\frac{2}{5}$$

The second derivative test tells us that since the second derivative of profit is negative (everywhere, but in particular at this point), the function is concave and hump-shaped, making this a maximum.

Recall that $P = 75 - \frac{1}{5}Q$. When quantity is 62.5, $P = 62.5$. Total revenue is $62.5 \times 62.5 = 3906.25$. Total cost is $50Q + 5000 = 625 + 5000 = 5625$. This yields profit $\Pi = 3906.25 - 5625 = -1718.75$. Because this is maximum profit, we know that producing any other quantity will yield profit more negative than this. Consequently, a competent project engineer will recommend that the firm not pursue this project unless costs can be lowered or the demand curve can be shifted to sell more units at a higher price.

## 2.5 Breakeven Quantity

Often, a firm will have the choice between pursuing one process with a high fixed cost and a low average variable cost and another process with a lower fixed cost but higher average variable costs. This can present a difficult choice for the firm, because at low quantities the low fixed cost option is more attractive, but at high quantities the low average variable cost will lead to long-term savings.

One useful analytic tool that we can use in order to help the firm determine which process to use is called the **breakeven quantity**. The purpose of the breakeven quantity is a bit different from our other optimization techniques. While profit maximization tells us what quantity to produce, breakeven quantity simply tells us which process to use once we know what quantity we would like to produce. We can use the breakeven quantity as extra information to help determine optimal profit.

In order to compare two processes, we need their cost functions. In the simplest case, we have two constant-AVC functions., $TC_1(Q) = AVC_1 \times Q + FC_1$ and $TC_2(Q) = AVC_2 \times Q + FC_2$. Assume that $AVC_1 > AVC_2$ but that $FC_1 < FC_2$. The breakeven quantity is the quantity $Q^*$ such that

$$TC_1(Q) = TC_2(Q)$$
$$AVC_1 \times Q^* + FC_1 = AVC_2 \times Q^* + FC_2$$
$$(AVC_1 - AVC_2)Q^* = FC_2 - FC_2$$
$$Q^* = \frac{FC_2 - FC_1}{AVC_1 - AVC_2}$$

The same method - set the two cost functions equal to find the *isocost point*[2] and solve for the quantity $Q^*$ at that point - is useful even if the cost function is not a constant-AVC function.

### 2.5.1 A Worked Example

**Problem.** A firm producing gadgets currently faces a constant average variable cost of 10 per gadget, including both materials and labor. Its fixed costs are 1000. The firm has the option of offering intensive training at a cost of 4000 in addition to its fixed costs of 1000. This training will allow workers to produce more efficiently, reducing variable costs to 7.5 per unit. At what predicted level of production does it make sense to schedule the new training?

**Solution.** Find the isocost point:

$$TC_1(Q^*) = TC_2(Q^*)$$
$$10Q^* + 1000 = 7.5Q^* + 5000$$
$$2.5Q^* = 4000$$

[2] The isocost point is the point where costs are the same.

$$Q^* = 1600$$

If the firm plans to produce at least 1600 units, it makes sense to schedule the more expensive training. If not, the firm will lose money on training.

Note also that when comparing constant-AVC cost functions, this equation reduces immediately to:

$$Q^* = \frac{\Delta FC}{\Delta AVC}$$

### 2.5.2 Nonlinear Cost Functions

As we saw from the previous example, breakeven quantity can be found relatively quickly using constant-AVC (linear) cost functions. The formula is useful when one process has a high average variable cost and low fixed cost, and the other has a low average variable cost and a high fixed cost. If one process has both lower fixed costs and a lower average variable cost, the lower-cost process should be preferred under all circumstances.

Sometimes it can be harder to "eyeball" which cost function increases faster or slower, even if you can figure out which one starts at a higher point due to the higher fixed cost. When you cannot determine whether one cost function is always cheaper than the other, utilize the breakeven method. If the only quantity yielded is negative, then one process is always cheaper than the other. If you find at least one positive breakeven quantity, test values on each side of the breakeven quantity to determine which process is cheaper in which region of the cost curve.

### 2.5.3 A Worked Example

**Problem.** The firm is considering using a new process that will require considerable on-the-job training for even experienced operators. Currently, the firm's cost function is $TC_1(Q) = Q^2 + 25Q + 150$. The new process has no fixed cost due to its high on-the-job component. The cost function of the new process would be $TC_2(Q) = Q^2 + 20Q + \frac{5000}{Q}$. Find the isocost point and solve for the breakeven quantity.

**Solution.** Begin by finding the isocost point:

$$Q^2 + 25Q + 150 = Q^2 + 20Q + \frac{5000}{Q}$$

$$5Q + 150 - \frac{5000}{Q} = 0$$

Multiply everything by $Q$ to get it out of the denominator:

$$5Q^2 + 150Q - 5000 = 0$$

Divide through by 5:

$$Q^2 + 30Q - 1000 = 0$$

Factor this equation:

$$(Q + 50)(Q - 20) = 0$$

This shows that we break even at $Q = -50$ or at $Q = 20$. Since we cannot produce a negative quantity, the breakeven quantity is $Q^* = 20$. At any quantity above 20, the new process will save the firm money. Confirm this by checking a value larger than 20:

$$TC_1(100) = 100^2 + 25(100) + 150 = 10,000 + 2500 + 150 = 12,650$$

$$TC_2(100) = 100^2 + 20(100) + \frac{5000}{100} = 10,000 + 2000 + 50 = 12050$$

Thus, as quantity increases, the new process is cheaper than the old process, even with the high initial cost.