Fundamentals of Calculus for ECO 321

Engineering Economics will involve taking several simple derivatives. This note[1] is designed to serve as a cookbook for any derivatives you might be required to take, as well as provide a brief theoretical grounding in derivatives as they apply to economics.

Important Concepts

A **function** is a mathematical relationship between two variables.[2] Typically, these variables are x and y. A function is defined as a relationship between an input variable (usually x) and an output variable (usually y) such that for every value of x, only one y is generated. For example, you may recall the slope-intercept form of a line, $y = mx + b$, where a line can be uniquely identified by its slope m and its y-intercept b. For every value of x, only one value of y is possible. Often, functions are written using the notation $f(x)$, which is pronounced "eff of x." This notation specifies that x is the input and y is the output.

In economics, we will often see other variables in place of x and y. Most commonly, quantity (Q) will be an input value for other functions, such a revenue, cost, and profit. For example, total cost is a function of quantity, and is often written as $TC(Q)$. A function can be written to call a specified input value, too - for example, fixed cost is defined as the total cost before producing any units at all, so fixed cost is defined as $TC(0)$ (or, "total cost at $Q = 0$").

A **correspondence** is a relationship between two variables. All functions are correspondences, but not all correspondences are functions. A correspondence that is *not* a function is $y = \sqrt{(x)}$. Because a number has two square roots, one

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[2]Functions need not be mathematical, so long as they follow the "one output per input" rule. For example, consider the "father" function, Father(Tom). Taking Tom as its input, only one output - the biological father of Tom - is produced. Thus, "father" is a function. "Son," however, is not, since one person may have more than one son, generating multiple output values for a single input.

positive and one negative, a single value of x will have two values of y (again, one that is positive and one that is negative).[3]

A function's **degree** is the highest power to which any variable is raised. For example, $f(x) = 3x+1$ is a function of *degree 1*, because it contains the variable x raised to the first power. Any function of degree 1 is a line. $g(x) = x^2+2x+3$ is a function of *degree 2*, because the highest power to which its only variable is raised is 2. Any function, even a line, is called a curve.

The **slope** of a line is defined as the vertical change over the horizontal change, $\frac{\text{rise}}{\text{run}}$. "Rise over run" is an easy way to remember that the slope is mathematically defined as $\frac{\Delta y}{\Delta x}$, or the change in output over the change in input. This provides an estimate of the rate of change of a function's output value. The higher the slope is, the faster the function changes value. Recall that for a line of the format $y = mx + b$, the slope is m. Take the two functions $y_1 = x + 5$, with slope 1, and $y_2 = 2x + 5$, with slope 2. For $x = 0$, $y_1 = 5$ and $y_2 = 5$. For $x = 1$, $y_1 = 6$ and $y_2 = 7$. y_2, which has a higher slope, increases at a faster rate for the same unit change in x.

Slopes tell us not just how fast a function is changing but whether the function is increasing or decreasing. If the slope of a line is positive, the function is increasing; a high positive slope means that a function is increasing quickly. If the slope is negative, the function is decreasing. It is possible for a function with positive value to have a negative derivative; consider the rate of change of gasoline in my car with respect to miles. My car has a 14-gallon gas tank and I typically get 30 miles per gallon (so 1 mile driven uses $\frac{1}{30}$ gallon of gas). That means that an estimate for the number of gallons of gas in my tank, g, is a function of the number of miles driven since my last fillup, d. If $d = 0$, $g = 14$. Thus, the function would be $g = -\frac{1}{30}d + 14$. Even though it is impossible to have negative gallons of gas in my car, this function has a negative slope.

Methods for Finding Derivatives

The **derivative** of a function is like slope, but a bit more complex. (Only a bit.) It measures the rate of change of a function - that is, how quickly the output value is growing or shrinking. The derivative is itself a function, written as $\frac{dy}{dx}$ or as $f'(x)$. That means that the derivative can produce a single value for any value of x. That single value tells us what the slope of the function is at point x. Since a positive slope indicates that the function is increasing, so does a positive derivative; a negative derivative indicates the function is decreasing at that point. There are several different ways to take a derivative.

The quick and dirty method is called a **linear approximation**. This involves choosing two points that are very close together (x_1 and x_2) and measuring the slope of the line between them: $\frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x}$. Unless the function is linear, this won't give you the exact derivative. However, if you do this multiple times, choosing points each time that are successively closer and closer together,

[3] Try using a graphing calculator or WolframAlpha to show this. Compare it to the graph of $y = ln(x)$ in order to see a true function with a similar shape.

you'll get a value that is closer and closer and closer to the derivative. This is called *taking a limit*;[4] doing it by hand is very time-consuming and you don't need to know anything about it.

There are a few useful rules that helps us avoid having to manually take limits. The first is that the derivative of a constant is 0. Imagine the function $y = 3$. (Yes, this is a function.) For any change in x, the change in y is 0, because y does not depend on x.

The second important rule is called the **power rule**. The power rule tells us that a function's derivative is related to its degree (or power). In order to take the derivative of a function, take the exponent from its variable, multiply by that exponent, and reduce the exponent by one. For example, take the function $f(x) = 3x^2$. Applying the power rule, $f'(x) = 2 \times 3x^{2-1} = 6x^1 = 6x$.

Now, try the power rule on a line that goes through the origin (i.e., the y-intercept is 0). Take the function $f(x) = 2x$. The exponent on the variable x is 1, so apply the power rule by reducing that exponent from 1 to 0, and multiplying the remaining portion by the exponent 1. $f'(x) = 1 \times 2x^0$. Since $x^0 = 1$ (as long as $x \neq 0$), this reduces to $f'(x) = 1 \times 2 \times 1 = 2$. Of course, since the derivative is the slope of a line, and this line has slope 2, the power rule simply confirms what we already knew.

The third important rule is called **linearity**. Derivatives are a *linear operation*, which simply means that if the function we want to take the derivative of is made up of multiple things added or subtracted, we can simply take the derivatives of each of those pieces and then add the derivatives together. Consider the function $f(x) = x^2 + 5x + 3$. That means that $f'(x) = \frac{d}{dx}x^2 + \frac{d}{dx}5x + \frac{d}{dx}3$. In English, this is the same as saying the derivative of a sum is the sum of its derivatives. Let's try that out.

Applying the power rule, the derivative of x^2 is $2x$. The derivative of $5x$ is 5. The derivative of 3 is 0. Thus, $f'(x) = 2x + 5$. That means that for the function $f(x) = x^2 + 5x + 3$, at any value of x, the slope of the function is $2x + 5$. The linearity property is useful in economics because functions are typically second-degree or lower, meaning you have to take at most three simple derivatives in order to find the derivative of a function.

A Worked Example

Problem. A firm has cost function $TC(q) = q^2 - 40q + 250$. Find the firm's marginal cost. When is total cost increasing? When is it decreasing?

Solution. First, note that when economists say "marginal," they are referring to a rate of change. Thus, "marginal cost" means "the (first) derivative of the total cost function." Since the function is $TC(q)$, we know that total cost

[4]Specifically, the derivative of a function is

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(TC) is a function and q is its input variable. Marginal cost is therefore the change in total cost per unit change in quantity produced. The following notations all refer to the same thing: $MC(q)$, for "marginal cost;" $\frac{d(TC(q))}{dq}$; $TC'(q)$. For simplicity, we will use $MC(q)$ throughout the remainder of this solution to refer to the derivative of the function and $\frac{d}{dq}$ to refer to the derivative of one of the pieces of the function.

Using the property of *linearity*, we can say that $MC(q) = \frac{d}{dq}q^2 - \frac{d}{dq}40q + \frac{d}{dq}250$.

Using the *power rule*, $\frac{d}{dq}q^2 = 2 \times q^{2-1} = 2 \times q^1 = 2q$, and $\frac{d}{dq}40q = 40$ because $40q$ is linear with a slope of 40. Using the rule that the derivative of a constant is 0, $\frac{d}{dq}250 = 0$.

Substituting our solutions above, $MC(q) = \frac{d}{dq}q^2 - \frac{d}{dq}40q + \frac{d}{dq}250 = 2q - 40$. Since q is a quantity produced, it is always positive or 0. As long as $2q < 40$, marginal cost will be negative and total cost is decreasing. Once $2q = 40$, marginal cost is positive and total cost begins increasing. Additionally, as we produce more units, q gets bigger, meaning our total cost increases as well. Thus, marginal cost is positive and total cost is increasing at quantities greater than or equal to 20.

The Second Derivative

The **second derivative** of a function is the derivative of the first derivative. Written as $f''(x)$ or $\frac{d^2y}{dx^2}$, the second derivative tells us something about the shape of the function. If the second derivative is negative, the function's rate of change is decreasing and the function is called **concave**; if the second derivative is positive, the function's rate of change is increasing and the function is called **convex**.

A Worked Example

Problem. A firm has cost function $TC(q) = q^2 - 40q + 250$. Is this firm's cost function convex or concave?

Solution. As established in the previous worked example, $TC'(q) = 2q - 40$. To find the function's concavity, we need to take the second derivative, which is $TC''(q)$, and also the first derivative of $TC'(q)$. That is, we need to take the derivative of $2q - 40$. Since $2q - 40$ is linear with slope 2, we know that its derivative is 2. Consequently, $TC''(q) = 2$. Since the second derivative is positive, total cost is a convex function.

Optimization

Optimization is the process of finding an optimum, or "best" point. The optimum can be a maximum or a minimum, depending on what you wish to do. Firms often wish to minimize costs, for example, or maximize profit. The function we wish to optimize is called the **objective function**. Optimizing functions is a simple exercise in taking two derivatives.

The First Derivative Test

Any point at which the first derivative of a function is 0 is called a **critical point**.

Every minimum and every maximum is a critical point.[5] To find an optimum, then, the first step is to find all critical points of the objective function. In many cases, there will be only one such point. If so, proceed directly to the second derivative test, *infra*, to determine whether it is a minimum, a maximum, or a stationary point.

If there is more than one such point, test each point. Put each value of x into the objective function and *evaluate* the function (that is, determine its value at each critical point). If seeking a maximum, choose the point with the highest output value for further evaluation; if seeking a minimum, choose the point with the lowest output value. Then, test that value using the second derivative test below.

It is possible for a function to have no minimum, no maximum, or no critical points at all. We will see some examples of each type of function below.

The Second Derivative Test

A point where a function has a zero first derivative and a negative second derivative is a local maximum. Referring to our earlier definitions, we could restate this by saying that a concave function achieves a maximum at its critical point. Consider the function $f(x) = -x^2 + 5x + 10$. Its first derivative is $f'(x) = -2x + 5$. $f'(x) = 0$ when $x = 2.5$. But is this point a minimum or a maximum?

One (quite tedious) method is to evaluate the function at two points, one on either side, and determine if the stationary point evaluates at a higher or lower value. At $x = 2.5$, $f(x) = 16.25$. Convenient points to test might be $x = 2$ and $x = 3$. $f(2) = 16$ and $f(3) = 16$, meaning that $f(2.5)$ is a higher value. This indicates that $x = 2.5$ is a maximum point.

A second method is to test the second derivative to see if the function is concave or convex at $x = 2.5$. To do this, take the second derivative and apply the definition above: a maximum point will have a negative second derivative

[5] This does not necessarily mean that every point critical point is an optimum. For example, at $x = 0$, the first derivative of $f(x) = x^3$ is 0, but that point is neither a minimum nor a maximum. It is what is called a **stationary point**.

and a minimum point will have a positive second derivative.[6] In this case, $f''(x) = \frac{d}{dx}(-2x+5) = -2$. Because the second derivative is negative, the function is concave, and its only critical point is a maximum. Graph this function on WolframAlpha or using a graphing calculator to confirm this.

A Worked Example

Problem. A firm has cost function $TC(q) = q^2 - 40q + 250$ and faces demand of $q = 520 - 2P$. Find the firm's profit-maximizing quantity and price.

Solution. Profit (Π) is defined as total revenue minus total cost. Total revenue is defined as price times quantity. Thus, our objective function is $\Pi(q) = Pq - TC(q) = Pq - q^2 + 40q - 250$. In order to optimize this function, we need to solve our demand function for q; rearranging it, our demand function is $P = 260 - \frac{1}{2}q$. Rewriting the profit function,

$$\Pi(q) = (260 - \frac{1}{2}q)q - q^2 + 40q - 250$$

$$\Pi(q) = 260q - \frac{1}{2}q^2 - q^2 + 40q - 250 = -\frac{3}{2}q^2 + 300q - 250$$

$$\frac{d\Pi(q)}{dq} = -3q + 300$$

$\frac{d\Pi(q)}{dq} = 0$ when $300 = 3q$, or when $q = 100$. In order to determine whether this is a minimum or a maximum, we need to take the second derivative, $\frac{d^2\Pi(q)}{dq^2} = \frac{d}{dq}(-3q + 300) = -3$. Because this second derivative is negative, $q = 100$ is a maximum.

In order to find the profit-maximizing price, we can then put $q = 100$ into the demand function. Since $P = 260 - \frac{1}{2}q$, the profit-maximizing price is $260 - \frac{100}{2} = 260 - 50 = 210$.

[6] If the second derivative is 0, you may either resort to evaluating points, as above, or continue taking derivatives until you achieve a nonzero derivative, and then apply the concavity definition depending on whether the previous derivative was even or odd. This will never come up in this course.

Problems

1. Compute the following derivatives:
   (a) $f_1(x) = x - 10$
   (b) $f_2(x) = 10x + 12$
   (c) $f_3(x) = 10x - 12$
   (d) $f_4(x) = x^2 + 10x + 12$
   (e) $f_5(x) = x^2 - 10x + 12$
   (f) $f_6(x) = -x^2 + 10x + 12$
   (g) $f_7(x) = -x^2 - 10x + 12$
   (h) $f_8(x) = \frac{1}{x}$ (*HINT:* $\frac{1}{x} = x^{-1}$.)
2. Find all the critical points of the function $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. Is each point a minimum, a maximum, or neither?
3. Find all the critical points of the function $f(x) = x^3$. Is each point a minimum, a maximum, or neither?
4. A firm's total cost function is $TC(q) = q^2 - 40q + 500$. Find the firm's marginal cost by taking the first derivative of total cost. Find the minimum point of total cost. Then, find the total cost at that minimum point.
5. A firm's total cost function is $TC(q) = 3q^2 - 100q + 400$. Is the cost function concave or convex?
6. A firm's profit function is $\Pi(q) = -3q^2 + 100q - 400$. Is the profit function concave or convex?

Solutions

1. (a) $f_1'(x) = 1$
   (b) $f_2'(x) = 10$
   (c) $f_3'(x) = 10$
   (d) $f_4'(x) = 2x + 10$
   (e) $f_5'(x) = 2x - 10$
   (f) $f_6'(x) = -2x + 10$
   (g) $f_7'(x) = -2x - 10$
   (h) $f_8'(x) = -\frac{1}{x^2} = x^{-2}$

2. $f'(x) = x^2 - 2x - 3 = 0$. Factoring, $(x-3)(x+1) = 0$, meaning the function has critical points at $x = 3$ and $x = -1$. The function's second derivative is $2x - 2$. $f''(-1) = -2 - 2 = -4$ and $f''(3) = 6 - 2 = 4$, meaning that the function reaches a local maximum at $x = -1$ and a local minimum at $x = 3$. This function has no global minimum or maximum because it grows without bound.

3. $f'(x) = 3x^2$, which equals 0 only when $x = 0$. At this point, $f''(x) = 6x = 0$, meaning we cannot easily tell whether it is a minimum or maximum, so evaluate the function on either side. For example, $f(-1) = -1$ and $f(1) = 1$, so this point is neither a minimum nor a maximum (it is a stationary point).

4. $MC(q) = 2q - 40$, meaning that the minimum total cost occurs when $2q = 40$ or $q = 20$. This is a minimum because $MC'(q) = 2 > 0$. $TC(20) = 400 - 800 + 500 = 100$.

5. $TC'(q) = 6q - 100$ and $TC''(q) = 6$. Since the second derivative is positive, this function is convex and is U-shaped.

6. $TC'(q) = -6q + 100$ and $TC''(q) = -6$. Since the second derivative is negative, this function is concave and is hump-shaped.